

# Accelerated Bose-Einstein condensates in a double-well potential

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Devices based on ultracold atoms moving in an accelerating optical lattice or double-well potential are a promising tool for precise measurements of fundamental physical constants as well as for the construction of sensors. Here, we carefully analyze the model of a couple of BECs separated by a barrier in an accelerated field and we show how the observable quantities, mainly the period of the beating motion or of the phase-shift, are related to the physical parameters of the model as well as to the energy of the initial state.

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Laser-cooled atoms have drawn a lot of attention as for potential applications to interferometry and high-precision measurements, from the determination of gravitational constants to geophysical applications [1–4], see also [5, 6] for a recent review. The idea of using cold atoms moving in an accelerating optical lattice [7–11] has open the field to multiple applications. For instance, by means of method proposed by Cladé *et al* [12] a precise measurement of the Earth’s gravitational acceleration constant  $g$  were performed [13, 14]; the obtained results had a very high precision and only a tiny discrepancy between  $g$  measured by a Raman interferometry on laser-cooled atoms and a classical gravimeter resulted, in fact the absolute relative uncertainty  $\Delta g/g$  turns out to be of order  $3 \times 10^{-9}$ .

More recently, a value for the constant  $g$  has been measured using ultracold strontium atoms confined in an amplitude-modulated vertical optical lattice [15], improving a previous result [16] by using a larger number of atoms and reducing the initial temperature of the sample. Determination of  $g$  has been obtained by measuring the frequency  $\nu_B$  of the Bloch oscillations of the atoms in the vertical optical lattice and recalling that  $\nu_B = mgd/2\pi\hbar$ , where  $m$  is the mass of the Strontium atom,  $\hbar$  is the Planck constant and  $d$  is the lattice period. Since Bloch oscillations only occur for an one-body particle in a periodic field and under the effect of a Stark potential then has been chosen, in the experiment above, a particular Strontium’s isotope  $^{88}\text{Sr}$ ; in fact the scattering length  $a_s$  of atoms  $^{88}\text{Sr}$  is very small and thus it can be assumed that the effects of the atomic binary interactions are negligible. The obtained value for the constant  $g$  was consistent with the previous one but was affected by a larger relative uncertainty of order  $6 \times 10^{-6}$ , because of a larger scattering in repeated measurements, mainly due to the initial position instability of the trap. Such a technique is also proposed to measure surface forces [17].

On the other side, new technologies enable the construction of simple coherent matter-wave beam splitter based on atom chips. These devices have been shown to be capable of trapping and guiding ultracold atoms on a

microscale; BECs can be efficiently created in such small devices and coherent quantum phenomena have been observed. Interferometers based on a microchip can be widely used as highly sensitive devices because they allow measurement of quantum phases. Technologically, chip-based atom interferometers promise to be very useful as inertial and gravitational field sensor provided that the quantum evolution of the matter waves is not perturbed by the splitting process. It has been seen that in such a device a BEC cloud up to  $10^5$  Rubidium-87 atoms can be split in two clouds inducing a double-well trapping potential phase-preserving [18]. By means of such a devices a measurement of the Earth’s acceleration constant  $g$  has been performed with relative uncertainty of order  $2 \times 10^{-4}$  [19]. It is well known that one of the most relevant physical effect in a double-well model is the so called *beating motion* between the two wells; hence, in principle one can measure the beating period of the BEC in an accelerated double-well potential and then obtain the value of the gravitational constant, as done for BECs in an accelerated optical lattice. However, we would remark that in the case of  $^{87}\text{Rb}$  isotopes the scattering length  $a_s$  is not small and thus binary interactions must be effectively taken into account if one want to relate the Earth’s gravitational constant  $g$  with the beating motion of the two BEC’s clouds between the two wells. Therefore, in order to improve the analysis of the experimental output it is necessary to have a more complete understanding of the underlying theory of two BECs separate by a asymmetrical barrier.

The aim of this paper is to provide a solid theoretical ground for a BECs in a double well potentials under the effect of the gravity force, where an explicit formula connecting the physical parameters, and in particular the Earth’s acceleration constant  $g$ , with the period of the observed beating motion between the two wells and of the difference of phase of two condensates. By means of such a result we expect that the relative uncertainty of the experimental results obtained for BECs in chips may be improved. Indeed, in such a framework the measure of the period of the difference of the phases between

the two condensates gives a precise value for the Earth's acceleration constant  $g$ . We would underline that our analysis will be useful even as a model for a.c. Josephson effects in BECs [20]

Here, we consider a simple model of BEC trapped in a double-well potential under the effect of a Stark potential, the dynamics along the direction of the gravity force is described by the one-dimensional Gross-Pitaevskii equation (GPE)

$$\begin{cases} i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V\psi + \epsilon|\psi|^2\psi + \nu x\psi \\ \psi(x, 0) = \psi_0(x) \end{cases}, \quad (1)$$

where  $V$  is the double-well trapping potential  $V$ ,  $\nu = mg$  is the strength of the Stark potential and the nonlinearity is given by  $\epsilon = \frac{4N\pi a_s \hbar^2}{m}$ , with  $N$  the total atom number,  $a_s$  is the scattering length,  $g$  is the gravity acceleration and  $m$  is the atom mass. The BEC wavefunction  $\psi$  is normalized to one.

By assuming the two-level approximation [21, 22] then the normalized BEC wave function  $\psi$  can be written as

$$\psi(x, t) = e^{-i\Omega t/\hbar} [a_R(t)\varphi_R(x) + a_L(t)\varphi_L(x)]$$

where  $a_{R,L}(t)$  are two complex valued functions depending on the time  $t$  satisfying

$$|a_R(t)|^2 + |a_L(t)|^2 = 1;$$

the vector  $\varphi_R$  (resp.  $\varphi_L$ ) corresponds to the ground state of the corresponding isolated right hand side (resp. left hand side) trap with associated energy  $\Omega$ .

It is well known that the solution to the unperturbed problem (1), where  $\epsilon = 0$  and  $\nu = 0$  and when the state is initially prepared on the first two ground states, exhibits a beating motion with period  $\frac{\pi\hbar}{\omega}$  independent of the initial wave function  $\psi_0$ , where  $\omega = \frac{E_- - E_+}{2}$  is half of the the splitting between the two onsite energies  $E_{\pm} = \Omega \mp \omega$ . Hence,  $\frac{\pi\hbar}{\omega}$  plays the role of *unit of time* and it is natural to introduce the (adimensional) *slow time*

$$\tau = \frac{\omega t}{\hbar}. \quad (2)$$

The amplitudes  $a_{R,L}(\tau)$  obey the nonlinear two-mode dynamical system given by (hereafter  $' = \frac{d}{d\tau}$ )

$$\begin{cases} ia'_R = -a_L + \eta|a_R|^2 a_R + \rho a_R \\ ia'_L = -a_R + \eta|a_L|^2 a_L - \rho a_L \end{cases}, \quad (3)$$

where  $\eta$  and  $\rho$  are the adimensional quantities defined as

$$\eta = \frac{\epsilon}{\omega} \int_{-\infty}^{+\infty} |\varphi_R(x)|^4 dx$$

and

$$\rho = \frac{\nu}{\omega} \int_{-\infty}^{+\infty} x |\varphi_R(x)|^2 dx.$$

The unperturbed solution (for  $\eta = \rho = 0$ ) of the two-level approximation has periodic solution with period  $T = \pi$ .

If we set  $a_{R,L}(\tau) = q_{R,L}(\tau)e^{i\theta_{R,L}(\tau)}$ , where  $q_{R,L} \in [0, 1]$  are such that  $q_R^2(\tau) + q_L^2(\tau) = 1$ , then the previous system (3) takes the Hamiltonian form

$$\begin{cases} \theta' = -\frac{\partial \mathcal{H}}{\partial \theta} \\ z' = \frac{\partial \mathcal{H}}{\partial z} \end{cases} \quad (4)$$

where  $\theta := \theta_R - \theta_L$  is the phase shift and  $z := q_R^2 - q_L^2$  is the imbalance function between the two condensates, with Hamiltonian function

$$\mathcal{H} = -2\sqrt{1-z^2} \cos \theta + \frac{1}{2}\eta(1+z^2) + 2\rho z. \quad (5)$$

We should remark that (4) is invariant with respect to the change of the sign of  $\epsilon$  and  $\rho$ ; more precisely, if  $\rho < 0$  then we can switch to the case of  $\rho > 0$  by changing the signs  $z \rightarrow -z$  and  $\theta \rightarrow -\theta$ . Similarly, if  $\eta < 0$  then we can switch to the case  $\eta > 0$  by  $z \rightarrow -z$  and  $\theta \rightarrow \theta + \pi$ .

It is a remarkable fact that equation (4) admits explicit periodical solutions, and that the period  $T$  of the imbalance function  $z(\tau)$ , as well as of the phase shift  $\theta(\tau)$ , can be explicitly computed as function of the parameters  $\eta$  and  $\rho$  as well as of the initial wave function [23]. Therefore, in principle, if one experimentally measure the inversion frequency then one can obtain a precise value for the acceleration constant  $g$ . In fact, let us denote by  $E$  the energy value of the Hamiltonian  $\mathcal{H}$  on the initial state:  $E := \mathcal{H}(z_0, \theta_0)$ . Then, from (4) and (5) we have that  $z(\tau)$  is a solution to the following ordinary differential equation of first order:

$$(z')^2 = az^4 + bz^3 + cz^2 + dz + e \quad (6)$$

where we set  $a = -\frac{1}{4}\eta^2$ ,  $b = -2\eta\rho$ ,  $c = E\eta - 4 - \frac{\eta^2}{2} - 4\rho^2$ ,  $d = 4E\rho - 2\eta\rho$  and  $e = E\eta - E^2 + 4 - \frac{1}{4}\eta^2$ . Equation (6) has solution given by means of the Weierstrass's elliptic function  $\mathcal{P}(\tau; g_2, g_3)$  with parameters

$$\begin{aligned} g_2 &:= ae - \frac{1}{4}bd + \frac{1}{12}c^2 \\ g_3 &:= -\frac{1}{16}eb^2 + \frac{1}{6}eac - \frac{1}{16}ad^2 + \frac{1}{48}dbc - \frac{1}{216}c^3 \end{aligned}$$

The Weierstrass's elliptic function  $\mathcal{P}(\tau; g_2, g_3)$  is a doubly periodic function which real period coincides with the period  $T$  of the phase shift and of the imbalance functions. In order to compute the real period  $T$  let  $e_j$ ,  $j = 1, 2, 3$ , be the roots of the trinomial  $4s^3 - g_2s - g_3$ ; and let  $\delta = g_2^3 - 27g_3^2$ . If  $\delta \geq 0$  then  $e_j \in \mathbb{R}$ ,  $e_3 < e_2 \leq 0 < e_1$ , and

$$T = \frac{2K(k)}{\sqrt{e_1 - e_3}}, \quad k = \frac{e_2 - e_3}{e_1 - e_3}$$

where  $K$  denotes the complete elliptic integral defined as

$$K(k) = \int_0^1 [(1-s^2)(1-ks^2)]^{-1/2} ds.$$

On the other side, if  $\delta < 0$  then  $e_2 \in R$  and  $e_3 = \bar{e}_1$ , with  $\Im e_1 \neq 0$ , and

$$T = \frac{2K(k)}{\sqrt{H_2}}, \quad k = \frac{1}{2} - \frac{3e_2}{4H_2}, \quad H_2 = \sqrt{2e_2^2 + e_1e_3}.$$

In particular, when the nonlinear interaction is negligible, that is  $\eta = 0$ , then the three solutions simply are  $e_1 = \frac{2}{3}(1 + \rho^2)$ ,  $e_2 = e_3 = -\frac{1}{3}(1 + \rho^2)$  and the period  $T$  actually does not depend on the initial wave function  $\psi_0$ , but it only depends on  $\rho$ :

$$T = \frac{\pi}{\sqrt{1 + \rho^2}}. \quad (7)$$

In particular, in the limit of  $\rho = 0$  we recover the unperturbed beating period  $T = \pi$ .

However, in general the period  $T$  depends on the initial wave-function, as well as on the two parameters  $\rho$  and  $\eta$ . In order to estimate the dependence of the beating period  $T$  from the initial state and from the parameter  $\eta$ , corresponding to the strength of the nonlinear term, we consider, at first, the case where the initial wave function  $\psi_0$  corresponds to a minimum value for the energy Hamiltonian  $\mathcal{H}$  defined by (5). As appears in Fig. 1 the period  $T$ , corresponding to the energy minimum, actually depends of  $\eta$  for fixed value of  $\rho$ ; for instance, the value of the period  $T$  at  $\rho = 0$  and  $\eta = 4.2$  is approximatively one half of the period  $T$  at  $\rho = 0$  and  $\eta = 0$ . Only for large value of  $\rho$  we have a good agreement between the values of  $T$  for different values of  $\eta$ .

We consider now the case where we fix the value of the adimensional parameters  $\rho$  and  $\eta$  and we compute the period  $T$  as function of the energy of the initial state. For argument's sake we perform two numerical experiments (see Fig. 2); in the first one we fix  $\rho = 0.5$  and  $\eta = 0.2$ , while in the second one we fix  $\rho = 2.5$  and  $\eta = 4.2$ . For small values of the two parameters  $\rho = 0.5$  and  $\eta = 0.2$  then the period  $T$  takes values (in the adimensional unit  $\tau$  defined by (2)) from  $T = 2.754$  at  $E = -2.117$  to  $T = 2.853$  at  $E = 2.358$ , that is the period  $T$  lies in an interval which length is around 1.7% of the mean value of  $T$ , the value of  $T$  corresponding to the minimum value of the energy is  $T = 2.754$ . On the other hand, for larger values of the two parameters  $\rho = 2.5$  and  $\eta = 4.2$  then the period  $T$  takes values from  $T = 0.681$  at  $E = 9.416$  to  $T = 1.706$  at  $E = -0.9586$ , in such a case we have that the period  $T$  lies in an interval which length is around 43% of the mean value of  $T$ , the value of  $T$  corresponding to the minimum value of the energy is  $T = 1.68$ . Hence, we can conclude that for some values of the parameters the period  $T$  can strongly depend on the initial state.

We close by giving the expression of the adimensional quantities  $\rho$  and  $\eta$  as function of physical parameters

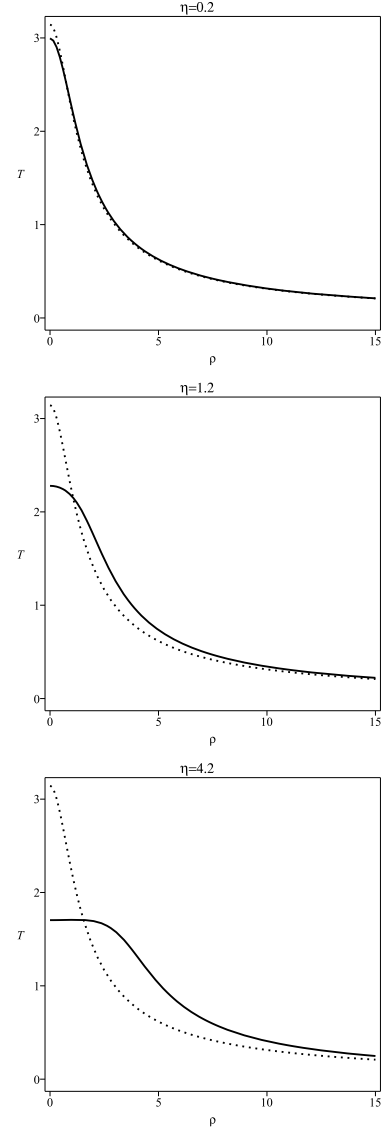


FIG. 1. In this figure we plot the graphs of the period  $T$  versus the adimensional parameter  $\rho$  for given values of  $\eta = 0.2$ ,  $\eta = 1.2$  and  $\eta = 4.2$  and when the initial stationary state  $\psi_0$  is associated to a minimum value for the energy Hamiltonian  $\mathcal{H}$  defined by (5). Broken line correspond to the limit case of  $\eta = 0$ , in such a case the period  $T$  is given by equation (7), and it does not depend on the initial wave function  $\psi_0$

in the semiclassical limit of small  $\hbar$ ; these formulas will be useful in order to compute the period  $T$  in a real device. To this end let us assume that the one-dimensional symmetric double-well potential is such that  $V(-x) = V(x)$  with two non-degenerate absolute minima points at  $x = \pm d$ , where  $2d$  is the distance between the bottom of the two wells, such that

$$V(\pm d) < 0, \quad \frac{dV(\pm d)}{dx} = 0 \quad \text{and} \quad \mu := \frac{d^2V(\pm d)}{dx^2} > 0,$$

we assume also that the double-well potential goes to zero

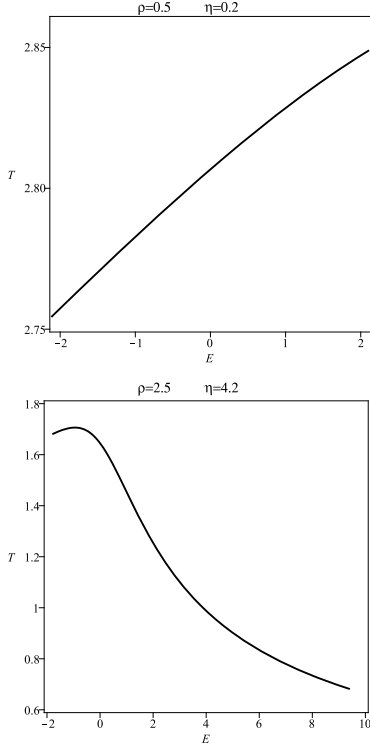


FIG. 2. In this figure we plot the graph of the period  $T$  versus the values of the energy  $E$  of the initial state for given values of  $\rho$  and  $\eta$ .

for large  $x$ . The eigenvalue equation  $-\frac{\hbar^2}{2m} \frac{d^2\varphi}{dx^2} + V\varphi = E\varphi$  admits two ground states  $E_{\pm} = \Omega \mp \omega$ , where  $\Omega = \frac{1}{2}\hbar\sqrt{\mu/m}[1 + O(\hbar)]$  is the ground state energy of the single trap in the limit of small  $\hbar$ , with associated normalized eigenvectors  $\varphi_{\pm}$ . These eigenvectors are even and odd-parity functions  $\varphi_{\pm}(-x) = \pm\varphi_{\pm}(x)$  and the normalized vectors  $\varphi_{R,L}$  are given by

$$\varphi_R = \frac{\varphi_+ + \varphi_-}{\sqrt{2}} \quad \text{and} \quad \varphi_L = \frac{\varphi_+ - \varphi_-}{\sqrt{2}}$$

By construction,  $\varphi_{R,L}(-x) = \varphi_{L,R}(x)$  and  $\varphi_R$  is mostly localized on one of the two well (say the well with center at  $x = +d$ ), and  $\varphi_L$  is mostly localized on the other well (say the well with center at  $x = -d$ ). In particular,

$$\varphi_R(x) = \varphi(x-d) \quad \text{and} \quad \varphi_L(x) = \varphi(x+d)$$

where  $\varphi(x)$  corresponds to the single trap eigenvector, that is

$$\varphi(x) = a(x; \hbar) e^{-\sqrt{m\mu}x^2/2\hbar}$$

with

$$a(x; \hbar) = \frac{(m\mu)^{1/8}}{(\pi\hbar)^{1/4}} (1 + O(\hbar))$$

in the semiclassical limit of small  $\hbar$ . Hence, if we denote by  $a_H = [\hbar^2/m\mu]^{1/4}$  the ground state oscillator length

then the leading term of the parameters  $\rho$  and  $\eta$  are given by

$$\rho = \frac{mgd}{\omega} \quad \text{and} \quad \eta = \frac{\epsilon}{\omega} \frac{1}{a_H \sqrt{2\pi}}$$

where  $\omega$  is half of the energy splitting,  $2d$  is the distance between the bottoms of the two wells and  $\epsilon$  is the strength of the Bose-Einstein condensate.

In conclusion: in this paper we have explored how relate the period  $T$  of the beating motion and of the phase-shift of a couple of accelerated condensates separated by a barrier with the physical parameters of the nonlinear asymmetrical double-well model, mainly the strength of the nonlinear term, the strength of the Stark potential (which breaks the symmetry of the double-well potential) and the splitting between the first two onsite energies. We have also seen that in general the period may be strongly dependent of the energy of the initial state. Therefore, experimental determination of the physical constants (typically the Earth's acceleration constant  $g$ ) should take into account such an effect.

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